# DIOPHANTINE ANALYSIS AROUND $[1, 2, 3, \dots]$

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Abstract. The transcendence of the regular infinite continued fraction $\mathfrak{z} := [1, 2, 3, 4, 5, \dots]$ was first proven by C.L.Siegel in 1929. The value of $\mathfrak{z}$ is a ratio of the values of modified Bessel functions. In this paper our diophantine analysis around $\mathfrak{z}$ takes its starting point with its rational convergents and deals with an asymptotic approximation formula for $\mathfrak{z}$ and with the construction of a sequence of quadratically irrational approximations using these convergents. Finally, we study various error sums for $\mathfrak{z}$ which are also defined by the rational convergents.

Dedicated to the memory of Professor Eduard Wirsing (1931 - 2022)

## 1. Introduction of the Zopf-number

In this paper we study a special number whose partial denominators form one of the simplest arithmetic sequences, namely $1, 2, 3, \dots$. Indeed, in 1929 Siegel [8] laid the groundwork for our study by treating the subject of our work as a special case among a family of quasi-periodic continued fractions as a ratio of modified Bessel functions of the first kind,

$$\frac{I_{a/b}(2/b)}{I_{a/b+1}(2/b)} = \left[\, \overline{a + kb}\, \right]_{k=0}^{\infty},$$

where $a, b \in \mathbb{Z}$ with $b > 0$ and $a + b > 0$. With the method later named after him, values of analytic functions which satisfy a linear differential equation and whose coefficients fulfill analytic and algebraic conditions in their Taylor expansion can be proved to be transcendental. In this paper we restrict our analytic and diophantine investigations only to the special continued fraction

$$\mathfrak{z} := 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \ddots}}} = [1, 2, 3, 4, 5, \dots].$$

In Section 2, we establish an asymptotic estimation around the error of approximation of $\mathfrak{z}$ by its rational convergents $p_n/q_n$. We then introduce a new concept, called the quadratic convergents of an irrational number $\xi$.

Let $\xi = [a_0, a_1, a_2, \dots]$ be an irrational number where $a_0 \geq 1$. For $n \geq 1$, the quadratic convergents are given by

$$\mathcal{Q}_n(\xi) := \left[\, \overline{a_0, a_1, \dots, a_{n-1}}\, \right],$$

with characteristic polynomial $q_{n-1}X^2 - (p_{n-1} - q_{n-2})X - p_{n-2} = 0$. Our results in Section 3 constitute very few of the myriad interesting properties of the quadratic convergents for the continued fraction

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$\mathfrak{z} = [\,\overline{k}\,]_{k=1}^{\infty}$, for which we will henceforth refer to as the "Zopf-number". The name, translating to "twist", has been adopted by the authors because of the twist of $\mathbb{N}$ around unity after taking $\mathcal{G}_n := \gcd(q_{n-1}, p_{n-1} - q_{n-2}, p_{n-2})$ through positive values of $n$, and for the way its linear and quadratic convergents weave around the real line at $\mathfrak{z}$. Specifically,

$$\{\mathcal{G}_n\}_{n=1}^{\infty} = 1,1,1,2,1,3,1,4,1,5,1,6,1,7,1,8,1,9,1,10,\ldots .$$

We then move into Section 4 and establish results on error sums of the Zopf-number in terms of Bessel functions.

## 2. Rational convergents of the Zopf-number

**Theorem 1.** *Let $p_n/q_n$ be the $n$-th convergent of the Zopf-number $\mathfrak{z}$. Then we have*

$$\left|\mathfrak{z} - \frac{p_n}{q_n}\right| \sim \frac{\ln\ln(q_n)}{q_n^2 \ln(q_n)} .$$

*Proof.* By setting $\mathfrak{z} = [a_0, a_1, a_2, \ldots]$, we have $a_n = n+1$ for $n \geq 0$. Then by the well known inequalities for convergents of an irrational number, we have

$$\frac{1}{(n+4)q_n^2} \leq \frac{1}{q_n(q_n+q_{n+1})} < \left|\mathfrak{z} - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} \leq \frac{1}{(n+2)q_n^2} . \tag{1}$$

From the recursion formula for the denominators $q_n$ of the convergents, we now get the following inequalities:

$$(n+2)q_n \leq q_{n+1} \leq (n+3)q_n \qquad (n \geq 0) , \tag{2}$$

since

$$(n+2)q_n = a_{n+1}q_n \leq q_{n+1} \leq a_{n+1}q_n + q_n = (1+a_{n+1})q_n = (n+3)q_n .$$

Using the induction principle and (2), lower and upper bounds for $q_n$ depending only on $n$ can be proved:

$$(n+1)! \leq q_n \leq (n+2)! \qquad (n \geq 0) . \tag{3}$$

For $n = 0$ the inequalities in (3) are true because of $1 \leq q_0 = 1 \leq 2$. Next, let (3) be true for some integer $n \geq 0$. Using both (2) and the induction hypothesis twice, we obtain on the one side,

$$(n+2)! = (n+2)(n+1)! \leq (n+2)q_n \leq q_{n+1} ,$$

and on the other side,

$$q_{n+1} \leq (n+3)q_n \leq (n+3)(n+2)! = (n+3)! .$$

Thus (3) is shown.
From *Stirling's formula* we have the asymptotic relation

$$\ln(n!) \sim n\ln(n) , \tag{4}$$

which follows from [1, Eq. 6.1.41]. From (3), if we first take logarithms and then multiply on both ends by $1 = \ln(n!)/\ln(n!)$, we have

$$\left(1 + \frac{\ln(n+1)}{\ln(n!)}\right)\ln(n!) \leq \ln(q_n) \leq \left(1 + \frac{\ln(n+1)(n+2)}{\ln(n!)}\right)\ln(n!) \qquad (n \geq 2) . \tag{5}$$

Here, both fractions inside the parentheses tend to zero when $n$ increases, which follows from (4). Combining (4) and (5), we find two functions $\varepsilon_1(n)$ and $\varepsilon_2(n)$, such that

$$\lim_{n\to\infty} \varepsilon_k(n) = 1 \qquad (k = 1,2) , \tag{6}$$

and

$$\varepsilon_1(n) n\ln(n) \leq \ln(q_n) \leq \varepsilon_2(n) n\ln(n) \qquad (n \geq 2) . \tag{7}$$

We now take logarithms of the inequalities in (7):

$$\ln\big(\varepsilon_1(n)\big) + \ln(n) + \ln\ln(n) \leq \ln\ln(q_n) \leq \ln\big(\varepsilon_2(n)\big) + \ln(n) + \ln\ln(n) \qquad (n \geq 3) . \tag{8}$$

In the next step, we combine the inequalities in (7) and (8) to form:

$$\frac{\varepsilon_1(n)\ln(n)}{\ln\big(\varepsilon_2(n)\big)+\ln(n)+\ln\ln(n)} \;\le\; \frac{\ln(q_n)}{n\ln\ln(q_n)} \;\le\; \frac{\varepsilon_2(n)\ln(n)}{\ln\big(\varepsilon_1(n)\big)+\ln(n)+\ln\ln(n)} \qquad (n\ge 3)\,.$$

Taking (6) into account, both the left and right-hand fraction tend to 1 for increasing $n$. This proves

$$\frac{\ln(q_n)}{n\ln\ln(q_n)} \sim 1\,. \tag{9}$$

Finally, we access the inequalities in (1) and rearrange them in an equivalent form:

$$\frac{\ln(q_n)}{n\ln\ln(q_n)}\cdot\frac{n}{n+4} < \frac{q_n^2\ln(q_n)}{\ln\ln(q_n)}\cdot\Big|\mathfrak{z}-\frac{p_n}{q_n}\Big| < \frac{\ln(q_n)}{n\ln\ln(q_n)}\cdot\frac{n}{n+2} \qquad (n\ge 3)\,.$$

By (9) we conclude that

$$\lim_{n\to\infty}\Big(\frac{q_n^2\ln(q_n)}{\ln\ln(q_n)}\cdot\Big|\mathfrak{z}-\frac{p_n}{q_n}\Big|\Big) = 1 \quad\Longleftrightarrow\quad \Big|\mathfrak{z}-\frac{p_n}{q_n}\Big| \sim \frac{\ln\ln(q_n)}{q_n^2\ln(q_n)}\,.$$

This completes the proof of the theorem. □

For the sequence of the numbers $q_n$, see A001053 in OEIS.

## 3. Quadratic convergents of the Zopf-number

Let

$$\xi := [a_0, a_1, a_2, \ldots]$$

be an irrational number greater than 1, such that $a_0 \ge 1$. Again, we denote the *(rational) convergents* of $\xi$ by $p_m/q_m$, for $m\ge 0$. Then,

$$\frac{1}{(2+a_{m+1})q_m^2} < \Big|\,\xi-\frac{p_m}{q_m}\,\Big| < \frac{1}{a_{m+1}q_m^2} \qquad (m\ge 0)\,. \tag{10}$$

Both inequalities in (10) result from inequalities (8) and (12) in [5, §13] and from the recurrence formula $q_{m+1} = a_{m+1}q_m + q_{m-1}$. Let the *quadratic convergents* of $\xi$ be given by

$$\mathcal{Q}_n(\xi) := [\,\overline{a_0, a_1, \ldots, a_{n-1}}\,]$$

for $n \ge 1$.

**Proposition 1.** *We have for every integer $n\ge 1$,*

$$\max\Big\{0,\Big(\frac{1}{2+a_n}-\frac{1}{a_0}\Big),\Big(\frac{1}{2+a_0}-\frac{1}{a_n}\Big)\Big\}\cdot\frac{1}{q_{n-1}^2} < \big|\xi-\mathcal{Q}_n(\xi)\big| < \Big(\frac{1}{a_0}+\frac{1}{a_n}\Big)\cdot\frac{1}{q_{n-1}^2}\,. \tag{11}$$

*Proof.* We have

$$\mathcal{Q}_n(\xi) = [\,a_0, a_1, \ldots, a_{n-1}, \mathcal{Q}_n(\xi)\,]\,.$$

Therefore, the rationals

$$\frac{p_0}{q_0},\ \frac{p_1}{q_1},\ldots,\ \frac{p_{n-1}}{q_{n-1}}$$

are convergents of $\xi$ as well as convergents of $\mathcal{Q}_n(\xi)$. Hence, we obtain four inequalities,

$$\frac{1}{(2+a_n)q_{n-1}^2} < \Big|\,\xi-\frac{p_{n-1}}{q_{n-1}}\,\Big| < \frac{1}{a_nq_{n-1}^2}\,, \tag{12}$$

$$\frac{1}{(2+a_0)q_{n-1}^2} < \Big|\,\mathcal{Q}_n(\xi)-\frac{p_{n-1}}{q_{n-1}}\,\Big| < \frac{1}{a_0q_{n-1}^2}\,. \tag{13}$$

We now link these inequalities twice with the triangle inequalities. On the one side, we have

$$\begin{aligned} \big|\xi - \mathcal{Q}_n(\xi)\big| &= \Big|\Big(\xi - \frac{p_{n-1}}{q_{n-1}}\Big) + \Big(\frac{p_{n-1}}{q_{n-1}} - \mathcal{Q}_n(\xi)\Big)\Big| \leq \Big|\,\xi - \frac{p_{n-1}}{q_{n-1}}\,\Big| + \Big|\,\mathcal{Q}_n(\xi) - \frac{p_{n-1}}{q_{n-1}}\,\Big| \\ &\overset{(12),(13)}{<} \Big(\frac{1}{a_0} + \frac{1}{a_n}\Big) \cdot \frac{1}{q_{n-1}^2}\,, \end{aligned} \tag{14}$$

and on the other side,

$$\begin{aligned} \big|\xi - \mathcal{Q}_n(\xi)\big| &= \Big|\Big(\xi - \frac{p_{n-1}}{q_{n-1}}\Big) + \Big(\frac{p_{n-1}}{q_{n-1}} - \mathcal{Q}_n(\xi)\Big)\Big| \\ &\geq \max\Big\{\Big|\,\xi - \frac{p_{n-1}}{q_{n-1}}\,\Big| - \Big|\,\mathcal{Q}_n(\xi) - \frac{p_{n-1}}{q_{n-1}}\,\Big|\,,\,\Big|\,\mathcal{Q}_n(\xi) - \frac{p_{n-1}}{q_{n-1}}\,\Big| - \Big|\,\xi - \frac{p_{n-1}}{q_{n-1}}\,\Big|\Big\} \\ &\overset{(12),(13)}{>} \max\Big\{\Big(\frac{1}{2+a_n} - \frac{1}{a_0}\Big)\frac{1}{q_{n-1}^2}\,,\,\Big(\frac{1}{2+a_0} - \frac{1}{a_n}\Big)\frac{1}{q_{n-1}^2}\Big\} \\ &= \max\Big\{\Big(\frac{1}{2+a_n} - \frac{1}{a_0}\Big)\,,\,\Big(\frac{1}{2+a_0} - \frac{1}{a_n}\Big)\Big\} \cdot \frac{1}{q_{n-1}^2}\,. \end{aligned} \tag{15}$$

(14) and (15) complete the proof of the Proposition. ☐

### 3.1. **An application to the Zopf-number.**

**Proposition 2.** *We have for every integer $n \geq 1$ with $|a_0 - a_n| \geq 3$,*

$$\big|\xi - \mathcal{Q}_n(\xi)\big| \;>\; \max\Big\{\frac{1}{(2+a_0)(3+a_0)}, \frac{1}{(2+a_n)(3+a_n)}\Big\} \cdot \frac{1}{q_{n-1}^2}\,. \tag{16}$$

**Corollary 1.** *For the Zopf-number $\mathfrak{z}$ we have the inequalities*

$$\frac{1}{12q_{n-1}^2} \;<\; \big|\mathfrak{z} - \mathcal{Q}_n(\mathfrak{z})\big| \;<\; \frac{5}{4q_{n-1}^2} \qquad (n \geq 3)\,. \tag{17}$$

*Proof of Corollary 1.* The number $\mathfrak{z}$ is such that $a_n = n+1$, and so $|a_0 - a_n| = n$. The lower bound in (17) then follows directly from (16), The upper bound in (17) is a consequence of the right inequality in (11). The latter is because of $n \geq 3$ and

$$\frac{1}{a_0} + \frac{1}{a_n} = \frac{n+2}{n+1} \leq \frac{5}{4}.$$

This proves the corollary. ☐

*Proof of Proposition 2.* We assume the condition $|a_0 - a_n| \geq 3$.
*Case 1.* Let $a_0 - a_n \geq 3$.
Then, we obtain

$$\begin{aligned} a_0 \;\geq\; 3 + a_n \;\iff\;& -\frac{1}{a_0} \;\geq\; -\frac{1}{3+a_n} \\ \iff\;& \frac{1}{2+a_n} - \frac{1}{a_0} \;\geq\; \frac{1}{2+a_n} - \frac{1}{2+a_n} \cdot \frac{2+a_n}{3+a_n} \;=\; \frac{1}{2+a_n} \cdot \frac{1}{3+a_n}\,. \end{aligned} \tag{18}$$

*Case 2.* Let $a_n - a_0 \geq 3$.
Interchanging $a_0$ and $a_n$ in Case 1, we obtain from (18),

$$\frac{1}{2+a_0} - \frac{1}{a_n} \geq \frac{1}{2+a_0} \cdot \frac{1}{3+a_0}\,. \tag{19}$$

Finally, (16) follows from (18), (19), and from the left inequality in (11) of Proposition 1. □

**Proposition 3.** *Let $n \geq 1$ with $a_n \geq 2 + a_0$. Then we have*

$$\xi - \mathcal{Q}_n(\xi) \begin{cases} > 0 & \textit{for} \quad n \equiv 0 \pmod 2\,, \\ < 0 & \textit{for} \quad n \equiv 1 \pmod 2\,. \end{cases} \tag{20}$$

*Proof.* $p_{n-1}/q_{n-1}$ is a common rational convergent of $\xi$ and $\mathcal{Q}_n(\xi)$. Depending on whether $n$ is odd or even, we obtain from (10) with $m = n-1$ and $a_0 = a_n$,

$$(-1)^{n-1}\mathcal{Q}_n(\xi) + (-1)^n \frac{p_{n-1}}{q_{n-1}} > \frac{1}{(2+a_0)q_{n-1}^2}\,. \tag{21}$$

Similarly, from (10) we obtain

$$\left|\xi - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{a_n q_{n-1}^2}\,. \tag{22}$$

Setting $Q(n) := (-1)^{n-1}\mathcal{Q}_n(\xi) + (-1)^n \xi$, this gives

$$\begin{aligned} Q(n) &= (-1)^{n-1}\mathcal{Q}_n(\xi) + (-1)^n \frac{p_{n-1}}{q_{n-1}} + (-1)^n \Big(\xi - \frac{p_{n-1}}{q_{n-1}}\Big) \\ &\geq (-1)^{n-1}\mathcal{Q}_n(\xi) + (-1)^n \frac{p_{n-1}}{q_{n-1}} - \Big|\xi - \frac{p_{n-1}}{q_{n-1}}\Big| \\ &\overset{(21),(22)}{>} \frac{1}{(2+a_0)q_{n-1}^2} - \frac{1}{a_n q_{n-1}^2} \geq \frac{1}{(2+a_0)q_{n-1}^2} - \frac{1}{(2+a_0)q_{n-1}^2} \;=\; 0\,. \end{aligned}$$

This proves the inequalities in (21) and completes the proof of the proposition. □

**Theorem 2.** *The quadratic convergents $\mathcal{Q}_n(\mathfrak{z})$ of the Zopf-number $\mathfrak{z}$ satisfy the inequalities*

$$\mathcal{Q}_2(\mathfrak{z}) < \mathcal{Q}_4(\mathfrak{z}) < \mathcal{Q}_6(\mathfrak{z}) < \cdots < \mathfrak{z} < \cdots < \mathcal{Q}_7(\mathfrak{z}) < \mathcal{Q}_5(\mathfrak{z}) < \mathcal{Q}_3(\mathfrak{z}) < \mathcal{Q}_1(\mathfrak{z})\,, \tag{23}$$

*where*

$$\lim_{n\to\infty} \mathcal{Q}_n(\mathfrak{z}) \;=\; \mathfrak{z}\,. \tag{24}$$

*Proof.* (24) follows immediately from Corollary 1, so it remains to prove (23). From the regular continued fraction expansion of the Zopf-number, we have that $a_n = n+1$ for $n \geq 0$. Thus, $a_n = n+1 \geq 3 = 2 + a_0$ is fulfilled for $n \geq 2$. Then, Proposition 3 yields

$$\mathcal{Q}_{2m}(\mathfrak{z}) \;<\; \mathfrak{z} \;<\; \mathcal{Q}_{2m+1}(\mathfrak{z}) \qquad (m \geq 1)\,. \tag{25}$$

Next, we prove that

$$\big|\mathfrak{z} - \mathcal{Q}_{n+1}(\mathfrak{z})\big| \;<\; \big|\mathfrak{z} - \mathcal{Q}_n(\mathfrak{z})\big| \qquad (n \geq 3)\,. \tag{26}$$

For this purpose, note that

$$4 \;\leq\; n+1 \;=\; a_n \;<\; [\,a_n, a_{n-1}, \ldots, a_1\,] \;=\; \frac{q_n}{q_{n-1}}\,.$$

Consequently, we have $q_n^2 > 16q_{n-1}^2 > 15q_{n-1}^2$, or

$$\frac{5}{4q_n^2} \;<\; \frac{1}{12q_{n-1}^2} \qquad (n \geq 3)\,.$$

Finally, we apply Corollary 1 twice, namely for $n+1$ and for $n$. This gives

$$\big|\mathfrak{z} - \mathcal{Q}_{n+1}(\mathfrak{z})\big| \,<\, \frac{5}{4q_n^2} \,<\, \frac{1}{12q_{n-1}^2} \,<\, \big|\mathfrak{z} - \mathcal{Q}_n(\mathfrak{z})\big| \qquad (n \geq 3)\,.$$

This proves (26). Combining (25) and (26), it turns out that

$$\mathcal{Q}_4(\mathfrak{z}) < \mathcal{Q}_6(\mathfrak{z}) < \cdots < \mathfrak{z} < \cdots < \mathcal{Q}_7(\mathfrak{z}) < \mathcal{Q}_5(\mathfrak{z}) < \mathcal{Q}_3(\mathfrak{z})\,.$$

In order to complete the proof of the theorem, we have to check the two inequalities

$$\mathcal{Q}_2(\mathfrak{z}) < \mathcal{Q}_4(\mathfrak{z})\,,$$
$$\mathcal{Q}_3(\mathfrak{z}) < \mathcal{Q}_1(\mathfrak{z})$$

numerically. We have

$$\mathcal{Q}_1(\mathfrak{z}) \;=\; \frac{1+\sqrt{5}}{2} \;=\; 1.618033\ldots\,, \quad \mathcal{Q}_2(\mathfrak{z}) \;=\; \frac{1+\sqrt{3}}{2} \;=\; 1.366025\ldots\,,$$

$$\mathcal{Q}_3(\mathfrak{z}) \;=\; \frac{4+\sqrt{37}}{7} \;=\; 1.440394\ldots\,, \quad \mathcal{Q}_4(\mathfrak{z}) \;=\; \frac{9+2\sqrt{39}}{15} \;=\; 1.432666\ldots\,.$$

The theorem is proven. □

Proposition 3 and Theorem 2 show that the approximation of $\mathfrak{z}$ with $\mathcal{Q}_n(\mathfrak{z})$ follows the same regularities as the approximation of any real irrational number with its rational convergents, c.f. (5) and (6) in [5, §13].

3.2. **Expressing quadratic convergents by numerators and denominators of rational convergents.** Let $\xi := [a_0, a_1, a_2, \ldots]$ be an irrational number greater than 1, such that $a_0 \geq 1$.

**Proposition 4.** *We have for all $n \geq 1$,*

$$\mathcal{Q}_n(\xi) \;=\; \frac{p_{n-1} - q_{n-2} + \sqrt{\left(p_{n-1} - q_{n-2}\right)^2 + 4q_{n-1}p_{n-2}}}{2q_{n-1}}\,, \tag{27}$$

*where $p_{-1} := 1$ and $q_{-1} := 0$.*

*Proof.* From $\mathcal{Q}_n(\xi) \,=\, [\,a_0, a_1, \ldots, a_{n-1}, \mathcal{Q}_n(\xi)\,]$ we obtain the identity

$$\mathcal{Q}_n(\xi) \;=\; \frac{p_{n-1}\mathcal{Q}_n(\xi) + p_{n-2}}{q_{n-1}\mathcal{Q}_n(\xi) + q_{n-2}} \qquad (n \geq 1)\,,$$

and therefore,

$$q_{n-1}\mathcal{Q}_n^2(\xi) - \big(p_{n-1} - q_{n-2}\big)\mathcal{Q}_n(\xi) - p_{n-2} \;=\; 0\,. \tag{28}$$

(27) now follows directly by applying the quadratic formula to (28). □

3.3. **On the approximation of the Zopf-number by quadratic convergents.**

For $\xi = \mathfrak{z}$, we know by (28) that $\mathcal{Q}_n(\mathfrak{z})$ is a root of the polynomial

$$P_n(X) \;:=\; q_{n-1}X^2 - (p_{n-1} - q_{n-2})X - p_{n-2}\,, \tag{29}$$

where $p_{-1} := 1$ and $q_{-1} := 0$. Since $7/5 < \mathfrak{z} < 3/2$ we have the inequalities

$$0 \,<\, \frac{7}{5}q_m \,<\, p_m \,<\, \frac{3}{2}q_m \qquad (m \geq 2)\,. \tag{30}$$

Furthermore we know from (2) in Section 2 that

$$(m+2)q_m \;\leq\; q_{m+1} \;\leq\; (m+3)q_m \qquad (m \geq 0)\,. \tag{31}$$

Denote the canonical height of a polynomial $P$ by $H(P)$ and the height of an algebraic number $\alpha$ by $H(\alpha)$.

**Lemma 1.** *The height of the polynomial $P_n$ from (29) is given by*

$$H\big(P_n\big) \;=\; p_{n-1} - q_{n-2} \qquad (n \geq 1)\,. \tag{32}$$

*Proof.* First, let $n \geq 3$.
We obtain from (30) (with $m = n-1 \geq 2$) and from (31) (with $m = n-2 \geq 1$),

$$p_{n-1} - q_{n-2} \;\overset{(31)}{\geq}\; p_{n-1} - \frac{q_{n-1}}{n} \;\overset{(30)}{>}\; \frac{7}{5}q_{n-1} - \frac{q_{n-1}}{n} \;=\; \Big(\frac{7}{5} - \frac{1}{n}\Big)q_{n-1} \;>\; q_{n-1}\,. \tag{33}$$

Additionally we have

$$q_{n-1} \;\overset{(31)}{\geq}\; nq_{n-2} \;\overset{(30)}{>}\; \frac{2}{3}np_{n-2} \;\geq\; p_{n-2}\,. \tag{34}$$

The assertion of the lemma for $n \geq 3$ follows from (33) and (34). Also for $n = 1$ and $n = 2$ the statement is correct, because

$$\begin{array}{lllllllll} H\big(P_1\big) &=& H\big(X^2 - X - 1\big) &=& 1 &=& 1-0 &=& p_0 - q_{-1}\,, \\ H\big(P_2\big) &=& H\big(2X^2 - 2X - 1\big) &=& 2 &=& 3-1 &=& p_1 - q_0\,. \end{array}$$

The lemma is proven. □

**Lemma 2.** *We have for all even integers $n \geq 2$,*

$$\mathcal{G}_n \;=\; \gcd\big(q_{n-1},\, p_{n-1} - q_{n-2},\, p_{n-2}\big) \;\equiv\; 0 \bmod \Big(\frac{n}{2}\Big)\,. \tag{35}$$

But it seems that much more is true.

**Conjecture 1.** *We have for all $n \geq 1$,*

$$\mathcal{G}_n \;=\; \begin{cases} 1 & \text{if} \quad n \equiv 1 \pmod 2\,, \\ n/2 & \text{if} \quad n \equiv 0 \pmod 2\,. \end{cases}$$

We will prove Lemma 2 below in Section 3.4.
Now, from Lemma 1 and 2 we have for every even integer $n \geq 2$,

$$H\big(\mathcal{Q}_n(\mathfrak{z})\big) \;\ll\; \frac{2\mathfrak{z}q_{n-1}}{n}\,, \tag{36}$$

since the difference $p_{n-1} - q_{n-2}$ in (32) has a simple asymptotic expansion

$$p_{n-1} - q_{n-2} \;=\; q_{n-1} \cdot \Big(\frac{p_{n-1}}{q_{n-1}} - \frac{q_{n-2}}{q_{n-1}}\Big) \sim \mathfrak{z}q_{n-1}\,, \tag{37}$$

and, consequently, $p_{n-1} - q_{n-2} \ll \mathfrak{z}q_{n-1}$. Note that we have the limits

$$\lim_{n\to\infty} \frac{p_{n-1}}{q_{n-1}} \;=\; \mathfrak{z}$$

and

$$\lim_{n\to\infty} \frac{q_{n-2}}{q_{n-1}} \;=\; \lim_{n\to\infty} [\,0, n, n-1, \ldots, 2\,] \;=\; 0\,.$$

For even $n$ the bound for $H\big(\mathcal{Q}_n(\mathfrak{z})\big)$ in (36) follows with Lemma 2 after multiplying (37) with $2/n$, so that $2(p_{n-1} - q_{n-2})/n \in \mathbb{Z}$.
We recall the inequalities from Corollary 1. The inequality in the following theorem results from the right-hand inequality in (17) and from (36).

**Theorem 3.** *We have for increasing even numbers $n$,*

$$\big|\,\mathfrak{z} - \mathcal{Q}_n(\mathfrak{z})\,\big| \;\ll\; \frac{1}{n^2 H^2\big(\mathcal{Q}_n(\mathfrak{z})\big)}\,. \tag{38}$$

In terms of evaluating the approximation quality of $|\mathfrak{z}-\mathcal{Q}_n(\mathfrak{z})|$ with the heights $H(\mathcal{Q}_n(\mathfrak{z}))$ the quadratic convergents $\mathcal{Q}_n(\mathfrak{z})$ approximate $\mathfrak{z}$ better than the rational numbers $p_n/q_n$ for even $n$, which admit only an approximation quality of the form

$$\Big|\mathfrak{z}-\frac{p_n}{q_n}\Big| \asymp \frac{1}{nq_n^2} \asymp \frac{1}{nH^2(p_n/q_n)}$$

(cf. formula (1) in Section 2), where $H(p_n/q_n)=p_n>q_n$, because $P_n(X)=q_nX-p_n$ and $\gcd(p_n,q_n)=1$.

Let $n$ be an even number. With formula (9) we can justify the second of the following inequalities:

$$n > n-1 \gg \frac{\ln(q_{n-1})}{\ln\ln(q_{n-1})}\,.$$

The function $\ln(x)/\ln\ln(x)$ is strictly increasing for $x\geq e^e=15.154\ldots$. Let $C>0$ be a constant in (36) such that $H\big(\mathcal{Q}_n(\mathfrak{z})\big)<2C\mathfrak{z}q_{n-1}/n$. Now, choosing $n$ sufficiently large, we have $e^e<H\big(\mathcal{Q}_n(\mathfrak{z})\big)<2C\mathfrak{z}q_{n-1}/n<q_{n-1}$, and obtain

$$n \gg \frac{\ln(2C\mathfrak{z}q_{n-1}/n)}{\ln\ln(2C\mathfrak{z}q_{n-1}/n)} > \frac{\ln\big(H(\mathcal{Q}_n(\mathfrak{z}))\big)}{\ln\ln\big(H(\mathcal{Q}_n(\mathfrak{z}))\big)}$$

for all large even $n$. This bound is now used to further enlarge the right-hand side in (38) by substituting for $n$.

**Corollary 2.** *For all large even integers $n$ we have*

$$\big|\mathfrak{z}-\mathcal{Q}_n(\mathfrak{z})\big| \ll \left(\frac{\ln\ln\big(H(\mathcal{Q}_n(\mathfrak{z}))\big)}{\ln\big(H(\mathcal{Q}_n(\mathfrak{z}))\big)\,H(\mathcal{Q}_n(\mathfrak{z}))}\right)^2.$$

### 3.4. **Proof of Lemma 2.**

**Lemma 3.** *Let $n\geq 2$ be an even integer. Then we have*

$$q_{n-1} \equiv \begin{cases} n/2 & (\mathrm{mod}\ n)\,, \quad \textit{if}\quad n\equiv 0 \pmod 4\,,\\ 0 & (\mathrm{mod}\ n)\,, \quad \textit{if}\quad n\equiv 2 \pmod 4\,. \end{cases} \tag{39}$$

$$p_{n-1} \equiv q_{n-2} \equiv \begin{cases} n/2+1 & (\mathrm{mod}\ n)\,, \quad \textit{if}\quad n\equiv 4 \pmod 8\,,\\ 1 & (\mathrm{mod}\ n)\,, \quad \textit{if}\quad n\equiv 0,2,6 \pmod 8\,. \end{cases} \tag{40}$$

$$p_{n-2} \equiv \begin{cases} 0 & (\mathrm{mod}\ n)\,, \quad \textit{if}\quad n\equiv 0 \pmod 4\,,\\ n/2 & (\mathrm{mod}\ n)\,, \quad \textit{if}\quad n\equiv 2 \pmod 4\,. \end{cases} \tag{41}$$

We prove this lemma in Section 3.5. But we can use Lemma 3 to complete the proof of Lemma 2: From (40) we conclude on $p_{n-1}-q_{n-2}\equiv 0 \pmod n$, which together with (39) and (41) implies Lemma 2. ☐

### 3.5. **Proof of Lemma 3.**

We demonstrate the arguments for the congruences in (40), the remaining congruences in (39) and (41) can be proven similarly. Before starting to prove (40), we need explicit formulas for $p_n$ and $q_n$, which have the form of sums over binomial coefficients.

**Lemma 4.** *We have for all $n\geq 0$,*

$$p_n = \sum_{k=0}^{\lfloor (n+1)/2\rfloor} (n-2k+1)!\binom{n-k+1}{k}^2\,, \tag{42}$$

$$q_n = \sum_{k=0}^{\lfloor n/2\rfloor} (n-2k)!\binom{n-k}{k}\binom{n-k+1}{k+1}\,. \tag{43}$$

*Proof.* Lemma 4 states a special case for Corollary 7 in [4]. □

With the numbers $p_i$ and $q_i$ for $i = 0,1,2,3$ obtained from Lemma 4, we check that (40) holds for $n = 2$ and $n = 4$. From now on we write $n'$ instead of $n/2$ for even integers $n \geq 6$.

Now we prove the two congruences in (40) for the numbers $p_{n-1}$ and $q_{n-2}$. First we treat $p_{n-1}$ using (42):

$$p_{n-1} = \sum_{k=0}^{n'} (n-2k)! \binom{n-k}{k}^2 =: \sum_{k=0}^{n'} c_k \tag{44}$$

with

$$c_k = \frac{(n-k)!}{k!}\binom{n-k}{k} = \frac{(n'-3)! \cdot (n'-2)(n'-1)n' \cdot (n'+1)\dots(n-k)}{k!}\binom{n-k}{k}. \tag{45}$$

*Case 1.1.* $0 \leq k \leq n'-3$. Note that $n \geq 6$ implies $n'-3 \geq 0$.
*(i)* By the condition of Case 1.1, $k!$ divides $(n'-3)!$.
*(ii)* Because of $(n'-2)(n'-1) \equiv 0 \pmod 2$ we have $(n'-2)(n'-1)n' \equiv 0 \pmod n$.
*(iii)* From (45) we find: $n-k \geq n'+3$.
Altogether we obtain

$$c_k \equiv 0 \pmod n. \tag{46}$$

*Case 1.2.* $k = n'-2$. Now $c_k$ has the form

$$c_{n'-2} = 4!\binom{n'+2}{n'-2}^2 = \frac{(n+4)^2(n+2)^2 n(n-2)^2}{3 \cdot 2^{11}}\, n. \tag{47}$$

*(i)* 3 divides $(n+4)(n+2)n$.
*(ii)* Exactly one of the four numbers $n+4$, $n+2$, $n$, $n-2$ is divisible by $2^3$, another by $2^2$, the remaining two are each divisible by 2. In total, $3 \cdot 2^{11}$ divides $(n+4)^2(n+2)^2 n(n-2)^2$. So for (47) we get

$$c_{n'-2} \equiv 0 \pmod n. \tag{48}$$

*Case 1.3.* $k = n'-1$.

$$c_{n'-1} = 2!\binom{n'+1}{n'-1}^2 = \frac{(n+2)^2 n}{2^5}\, n. \tag{49}$$

*Case 1.3.1.* $n \equiv 4 \pmod 8$. Then, $2^4$ divides $(n+2)^2 n$, but this does not hold for $2^5$. Thus, (49) yields

$$c_{n'-1} \equiv \frac{n}{2} \pmod n. \tag{50}$$

*Case 1.3.2.* $n \equiv 0,2,6 \pmod 8$. Under this assumption, $2^5$ divides $(n+2)^2 n$, and so we get for (49)

$$c_{n'-1} \equiv 0 \pmod n. \tag{51}$$

*Case 1.4.* $k = n'$.

$$c_{n'} = 0!\binom{n'}{n'} = 1. \tag{52}$$

Finally, (40) for $p_{n-1}$ follows from (44), (46), (48), (50), (51), and (52).

The following considerations will show that the same congruences hold for $q_{n-2}$. From formula (43) we obtain

$$q_{n-2} = \sum_{k=0}^{n'-1} (n-2k-2)! \binom{n-k-2}{k}\binom{n-k-1}{k+1} =: \sum_{k=0}^{n'-1} d_k. \tag{53}$$

*Case 2.1.* $0 \le k \le n' - 3$. We have

$$\begin{aligned} d_k &= \frac{(n-k-2)!}{k!} \binom{n-k-1}{k+1} \\ &= \frac{(n'-3)! \cdot (n'-2)(n'-1)n' \cdot (n'+1) \cdots (n-k-2)}{k!} \binom{n-k-1}{k+1}. \end{aligned} \tag{54}$$

Note that $n - k - 2 \ge n' + 1$. As above in Case 1.1, the hypothesis of Case 2.1 guarantees

$$d_k \equiv 0 \pmod{n}. \tag{55}$$

*Case 2.2.* $k = n' - 2$. We obtain

$$d_{n'-2} = 2! \binom{n'}{n'-2} \binom{n'+1}{n'-1} = \frac{(n-2)n(n+2)}{2^5} n. \tag{56}$$

*Case 2.2.1.* $n \equiv 4 \pmod 8$. Although $2^4$ divides $(n-2)n(n+2)$, this does not hold for $2^5$. Thus, (56) yields

$$d_{n'-2} \equiv \frac{n}{2} \pmod{n}. \tag{57}$$

*Case 2.2.2.* $n \equiv 0, 2, 6 \pmod 8$. Now, $2^5$ divides $(n-2)n(n+2)$. We obtain from (56),

$$d_{n'-2} \equiv 0 \pmod{n}. \tag{58}$$

*Case 2.3.* $k = n' - 1$. This results in

$$d_{n'-1} = 0! \binom{n'-1}{n'-1} \binom{n'}{n'} = 1. \tag{59}$$

Finally, (40) for $q_{n-2}$ follows from (53), (55), (57), (58), and (59).

This completes the proof of (40) in Lemma 3. □

## 4. Error sums of the Zopf-number

4.1. **Preliminaries.** Let $a \ge 0$ and $b \ge 1$ be integers. We define the numbers

$$\mathfrak{z}_a^b := a + 1 + \frac{b|}{|a+2} + \frac{b|}{|a+3} + \frac{b|}{|a+4} + \dots \tag{60}$$

by their irregular ($b > 1$) or regular ($b = 1$) continued fraction expansion. In particular, $\mathfrak{z}_a^1 = [a+1, a+2, a+3, \dots]$, and $\mathfrak{z} = \mathfrak{z}_0^1 = [1, 2, 3, \dots]$ gives the Zopf-number. The numbers

$$\mathcal{E}\big(\mathfrak{z}_a^b\big) := \sum_{m=0}^{\infty} (-1)^m \big(\mathfrak{z}_a^b q_m - p_m\big) \tag{61}$$

$$\mathcal{E}^*\big(\mathfrak{z}_a^b\big) := \sum_{m=0}^{\infty} \big(\mathfrak{z}_a^b q_m - p_m\big) \tag{62}$$

$$\mathcal{E}_{fac}\big(\mathfrak{z}_a^b\big) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \big(\mathfrak{z}_a^b q_m - p_m\big) \tag{63}$$

$$\mathcal{E}_{fac}^*\big(\mathfrak{z}_a^b\big) := \sum_{m=0}^{\infty} \frac{1}{m!} \big(\mathfrak{z}_a^b q_m - p_m\big) \tag{64}$$

are called *error sums* of $\mathfrak{z}$, where $p_m/q_m$ are the convergents obtained from the regular or irregular continued fraction of $\mathfrak{z}_a^b$. Both, the numerators $p_m$ and denominators $q_m$ satisfy the recurrence relation

$$u_{m+2} = (m+a+3)u_{m+1} + bu_m \qquad (m \ge 0) \tag{65}$$

with

$$\left.\begin{array}{lllllll} p_0 & = & a+1\,, & & q_0 & = & 1\,, \\ p_1 & = & a^2+3a+b+2\,, & & q_1 & = & a+2\,, \\ p_2 & = & a^3+6a^2+2ab+11a+4b+6\,, & & q_2 & = & a^2+5a+b+6\,, \end{array}\right\} \tag{66}$$

see [5, § 2, (12)]. Let us first consider two general types of error sums, namely the $\mathcal{E}$ and the $\mathcal{E}^*$ function:

$$\mathcal{E}^*(\xi) := \sum_{m=0}^{\infty} \big(\xi q_m - p_m\big)\,, \quad \mathcal{E}(\xi) := \sum_{m=0}^{\infty} \big|\xi q_m - p_m\big|\,. \tag{67}$$

The sums extend in each case over all convergents $p_m/q_m$ of a real number $\xi$. The theory of these two functions shows us that they have a fractal appearance, [2], [3]. The function $\mathcal{E}^*$ was first studied in more detail by *J.N.Ridley* and *G.Petruska* in 2000, [7]. In 1992 Petruska had already used a special irrational number and its errorsum $\mathcal{E}^*$ to explicitly construct a so-called $q$-series with a given radius of convergence greater than 1, [6].

We list some properties of these error sums.

1.) $\mathcal{E}(\xi)$ and $\mathcal{E}^*(\xi)$ are continuous functions at every irrational point $\xi$, and discontinuos functions at every rational point $\xi$.
2.) The range of the function $\mathcal{E}$ is the set of all real numbers between zero and the *Golden Number* $G = (1+\sqrt{5})/2$, whereas the range of the function $\mathcal{E}^*$ is the set of all real numbers between 0 and 1.
3.) Both functions are periodic with period 1.
4.) The error sums satisfy simple functional equations,

$$\mathcal{E}^*(\xi)+\mathcal{E}^*(1-\xi) = \begin{cases} 1-\xi & \text{if } 0<\xi<1/2\,, \\ \xi & \text{if } 1/2<\xi<1\,, \end{cases}$$

$$\mathcal{E}(\xi)-\mathcal{E}(1-\xi) = \begin{cases} \xi-1 & \text{if } 0<\xi<1/2\,, \\ \xi & \text{if } 1/2<\xi<1\,. \end{cases}$$

5.) From 1.) and 2.) it follows that $\mathcal{E}(\xi)$ and $\mathcal{E}^*(\xi)$ are Lebesgue integrable. We have

$$\int_0^1 \mathcal{E}^*(\alpha)\,d\alpha = \frac{3}{8}\,,$$

$$\int_0^1 \mathcal{E}(\alpha)\,d\alpha = \frac{3\zeta(2)\ln 2}{2\zeta(3)} - \frac{5}{8} = \frac{\pi^2\ln 2}{4\zeta(3)} - \frac{5}{8} = 0.79778798\ldots\,.$$

The functions

$$f(x) := \sum_{m=0}^{\infty} (\mathfrak{z}_a^b q_m - p_m)\cdot x^m \tag{68}$$

and

$$g(x) := \sum_{m=0}^{\infty} (\mathfrak{z}_a^b q_m - p_m)\cdot \frac{x^m}{m!} \tag{69}$$

are called *ordinary generating functions* and *exponential generating functions*, respectively, of the errors $\mathfrak{z}_a^b q_m - p_m$.

From here we make use of Bessel functions, where $\mu \geq 0$ is an integer.

$$\begin{array}{lll} J_\mu(x) & : & \textit{Bessel function of the first kind}\,, \\ Y_\mu(x) & : & \textit{Bessel function of the second kind}\,, \\ I_\mu(x) & : & \textit{modified Bessel function of the first kind}\,, \\ K_\mu(x) & := & \Gamma(\mu+1)\Big(\dfrac{x}{2}\Big)^{-\mu} J_\mu(x)\,. \end{array}$$

For example, we have

$$\mathfrak{z}_a^b \;=\; \sqrt{b}\cdot\frac{I_a(2\sqrt{b})}{I_{a+1}(2\sqrt{b})}\,. \tag{70}$$

For real numbers $t$ tending to infinity, we have the asymptotic behavior

$$J_\mu(t) \;\sim\; \sqrt{\frac{2}{\pi t}}\cos\Big(t-\frac{\mu\pi}{2}-\frac{\pi}{4}\Big)\,. \tag{71}$$

The following relationships between Bessel functions will play an essential role in our investigations.

**Lemma 5.** (i) *Let $x\in\mathbb{C}$. The relationship between $I_\mu(x)$ and $J_\mu(x)$ is given by*

$$I_\mu(x) \;=\; e^{-i\pi\mu/2}J_\mu\big(e^{i\pi/2}x\big)\,. \tag{72}$$

(ii) *We have the recurrence relation*

$$\frac{2\mu}{x}I_\mu(x) \;=\; I_{\mu-1}(x)-I_{\mu+1}(x) \qquad (\mu\geq 1)\,. \tag{73}$$

(iii) *Let $\beta,x\in\mathbb{C}$. Then we have*

$$\frac{d}{dx}\Big(x^{-\mu}I_\mu(\beta x)\Big) \;=\; \beta x^{-\mu}I_{\mu+1}(\beta x)\,. \tag{74}$$

(iv) *We have for $\mu\geq 1$,*

$$I_\mu'(x) \;=\; \frac{1}{2}\big(I_{\mu+1}(x)+I_{\mu-1}(x)\big)\,. \tag{75}$$

*Proof.* See [1], [10]. □

4.2. **Main Results.** For the error sums of $\mathfrak{z}_a^b q_m - p_m$, the exponential generating function $g(x)$ from (69) turns out to be the theoretically more accessible object than the ordinary generating function $f(x)$ in (68), albeit with the restriction to $a=0$. However, $\mathfrak{z}_0^b$ and $\mathfrak{z}_a^b$ are related only by one explicitly given linear fractional transformtion, namely

$$\mathfrak{z}_a^b \;=\; \frac{b\mathfrak{z}_0^b q_{a-2}-bp_{a-2}}{p_{a-1}-\mathfrak{z}_0^b q_{a-1}}\,,$$

and therefore the restriction to $a=0$ is immaterial. So we start by listing results for $g(x)$ for the error sums formed with $\mathfrak{z}_0^b$.

Formula numbers marked with an asterisk refer to formulas obtained using MAPLE. For the sake of brevity, we also omit the proofs of these formulas that can be done with known standard methods of real analysis. All other unmarked formula numbers refer to statements that are either evident or proven.

**Theorem 4.** *Let $a=0$ and $b\geq 1$. The function $g(x)$ satisfies the differential equation*

$$(x-1)g''+3g'+bg \;=\; 0\,. \tag{76}$$

*Moreover, we have*

$$g(x) \;=\; \sqrt{b}\,\frac{I_2(2\sqrt{b(1-x)})}{(1-x)I_1(2\sqrt{b})} \qquad (x\in\mathbb{R}\setminus\{1\})\,, \tag{77*}$$

$$\mathcal{E}_{fac}^*(\mathfrak{z}_0^b) \;=\; \lim_{x\to 1} g(x) \;=\; \frac{\sqrt{b^3}}{2I_1(2\sqrt{b})}\,, \tag{78}$$

$$\mathcal{E}_{fac}(\mathfrak{z}_0^b) \;=\; g(-1) \;=\; \sqrt{b}\,\frac{I_2(2\sqrt{2b})}{2I_1(2\sqrt{b})}\,. \tag{79}$$

(78) follows from (77) using the limit

$$\lim_{x\to 1} \frac{I_2(2\sqrt{b(1-x)})}{1-x} \;=\; \frac{b}{2}\,,$$

which can be obtained by replacing $x$ in

$$I_\mu(x) \;=\; \Big(\frac{x}{2}\Big)^\mu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(\mu+k+1)} \Big(\frac{x}{2}\Big)^{2k}$$

by $2\sqrt{b(1-x)}$.

**Corollary 3.** *For $b = 1$ we have*

$$\mathcal{E}^*_{fac}(\mathfrak{z}) \;=\; \frac{1}{2I_1(2)} \;=\; 0.314339\,\dots\,, \tag{80}$$

$$\mathcal{E}_{fac}(\mathfrak{z}) \;=\; \frac{I_2(2\sqrt{2})}{2I_1(2)} \;=\; 0.583891\,\dots\,. \tag{81}$$

The exponential generating function $g(x)$ on the right side in (69) is a Taylor expansion around the point $x = 0$. However, the expansion of this function around the point infinity is also particularly interesting. We give this expansion in the following theorem.

**Theorem 5.** *For positive increasing $x > 1$ we have the asymptotic expansion*

$$g(x) \;=\; \sqrt{\frac{\sqrt{b}}{\pi}} \cdot \Big( \frac{\sin\big(5\pi/4 + 2\sqrt{b(x-1)}\big)}{I_1(2\sqrt{b})x^{5/4}} + \frac{15}{16} \cdot \frac{\cos\big(5\pi/4 + 2\sqrt{b(x-1)}\big)}{I_1(2\sqrt{b})x^{7/4}} \Big) + \mathcal{O}\big(x^{-9/4}\big)\,. \qquad (82)^*$$

*For $b = 1$ there are rational numbers $q_{m,1}$ and $q_{m,2}$ such that we have for all real numbers $x > 1$ the series*

$$g(x) \;=\; \frac{1}{\sqrt{\pi}I_1(2)} \sum_{m=1}^{\infty} \Big( \frac{q_{m,1}}{x^{m+1/4}} \sin\Big(\frac{5\pi}{4} + 2\sqrt{x-1}\Big) \;+\; \frac{q_{m,2}}{x^{m+3/4}} \cos\Big(\frac{5\pi}{4} + 2\sqrt{x-1}\Big)\Big)\,. \qquad (83)^*$$

*In particular, $q_{1,1} = 1$ and $q_{1,2} = 15/16$.*

We obtain a corollary from Theorem 4 and Theorem 5.

**Corollary 4.** *We have*

$$\lim_{x\to\infty} g(x) \;=\; \lim_{x\to\infty} \sum_{m=0}^{\infty} (\mathfrak{z}_0^b q_m - p_m)\frac{x^m}{m!} \;=\; 0\,, \tag{84}$$

$$\int_0^\infty g(x)\,dx \;=\; \lim_{x\to\infty} \sum_{m=1}^{\infty} (\mathfrak{z}_0^b q_{m-1} - p_{m-1})\frac{x^m}{m!} \;=\; 1\,. \tag{85}$$

*Moreover, for every $\varepsilon > 0$ and sufficiently large $x > 1$ we have*

$$\big|x^{5/4}g(x)\big| \;<\; \frac{\sqrt[4]{b}}{\sqrt{\pi}} \cdot \frac{1}{I_1(2\sqrt{b})} + \varepsilon\,, \tag{86}$$

*where the numerical constant*

$$\frac{\sqrt[4]{b}}{\sqrt{\pi}} \cdot \frac{1}{I_1(2\sqrt{b})} \tag{87}$$

*is best-possible.*

Now we turn to the function $f(x)$. Again, we allow arbitrary integers $a \geq 0$ and $b \geq 1$ and we state our results for $f(x)$ in Theorem 6 and Theorem 7 without proofs. Also for these theorems only standard methods of real analysis are used.

**Theorem 6.** *The function $f(x)$ satisfies the differential equation*

$$x^2 f' + \big(bx^2 + (a+2)x - 1\big) f \;=\; bx - \mathfrak{z}_a^b + a + 1\,.$$

*Moreover, we have*

$$f(x) \;=\; \frac{1}{x^{a+2}e^{bx+1/x}} \Big( \int_1^x \big(bt - \mathfrak{z}_a^b + a + 1\big) t^a e^{bt+1/t}\,dt + e^{b+1}\mathcal{E}^*(\mathfrak{z}_a^b) \Big) \qquad (x > 0)$$

*and*

$$f(x) \;=\; \frac{1}{x^{a+2}e^{bx+1/x}} \Big( -\int_x^{-1} \big(bt - \mathfrak{z}_a^b + a + 1\big) t^a e^{bt+1/t}\,dt + (-1)^a e^{-b-1}\mathcal{E}(\mathfrak{z}_a^b) \Big) \qquad (x < 0)\,.$$

The function $e^{1/t}$ is analytical for every real number $t$ except $t = 0$. Therefore, we have in Theorem 6 no common formula for all real numbers $x$. For $x = 0$, the limit is given by

$$f(0) \;=\; \lim_{\varepsilon\to 0^+} -\frac{1}{\varepsilon^{a+2}e^{b\varepsilon+1/\varepsilon}} \Big( \int_\varepsilon^1 \big(bt - \mathfrak{z}_a^b + a + 1\big) t^a e^{bt+1/t}\,dt \Big) \;=\; \mathfrak{z}_a^b q_0 - p_0 \;=\; \mathfrak{z}_a^b - a - 1\,.$$

We end the listing of results with a theorem for an exponential sum, which becomes an *error sum for minor convergents* of $\mathfrak{z}_0^b$ when $b = 1$.

**Theorem 7.** *Let $a = 0$, $b \geq 1$, and $k \in \mathbb{N}$. Moreover, let $P_{k,m} := kp_{m+1} + p_m$ and $Q_{k,m} := kq_{m+1} + q_m$, where $p_m/q_m$ are the convergents of $\mathfrak{z}_0^b$ given by (65) and (66). Then we have*

$$h(k) \;:=\; \sum_{m=0}^{\infty} \frac{(k+1)^m}{m!} \cdot (\mathfrak{z}_0^b Q_{k,m} - P_{k,m}) \;=\; \frac{b}{i\sqrt{k}} \cdot \frac{I_2'(2i\sqrt{bk})}{I_1(2\sqrt{b})}\,,$$

$$\int_0^k h(t)\,dt \;=\; \sqrt{b} \cdot \frac{J_2(-2\sqrt{bk})}{I_1(2\sqrt{b})}\,.$$

*Example.* For $k = 1$ we obtain from Theorem 7,

$$\sum_{m=0}^{\infty} \frac{2^m}{m!} \cdot \big(\mathfrak{z}_0^b(q_{m+1} + q_m) - (p_{m+1} + p_m)\big) \;=\; \frac{bI_2'(2i\sqrt{b})}{iI_1(2\sqrt{b})} \;=\; \frac{b\big(I_1(2i\sqrt{b}) + I_3(2i\sqrt{b})\big)}{2iI_1(2\sqrt{b})}$$

$$=\; \frac{b\big(J_1(2\sqrt{b}) - J_3(2\sqrt{b})\big)}{2I_1(2\sqrt{b})}\,.$$

In the second last step, we first apply formula (75) with $\mu = 2$, and then (72) in order to obtain the final real expression.

4.3. **Proofs.** We prove the outstanding parts in this Section (apart from Theorems 6 and 7), skipping the intermediate computations performed with MAPLE. In some places, however, we give individual intermediate steps in the procedure even for the MAPLE calculations.

*Proof of (76) and (77) in Theorem 4.* Let

$$y(x) \;:=\; \sum_{m=0}^{\infty} \frac{u_m}{m!}\, x^m\,, \tag{88}$$

where the integers $u_m$ are given recursively by (65) and (66) with $a = 0$. Differentiating twice, we obtain the series

$$y'(x) \;=\; \sum_{m=1}^{\infty} \frac{u_m}{(m-1)!}\, x^{m-1} \quad \text{and} \quad y''(x) \;=\; \sum_{m=2}^{\infty} \frac{u_m}{(m-2)!}\, x^{m-2}\,. \tag{89}$$

With an index shift on the right of (89) and application of the recursion formula (65), we obtain

$$\begin{aligned} y''(x) &= \sum_{m=0}^{\infty} \frac{u_{m+2}}{m!}\, x^m = \sum_{m=2}^{\infty} \frac{u_m}{(m-2)!}\, x^{m-1} + 3\sum_{m=1}^{\infty} \frac{u_m}{(m-1)!}\, x^{m-1} + b\sum_{m=0}^{\infty} \frac{u_m}{m!}\, x^m \\ &= xy''(x) + 3y'(x) + by(x)\,. \end{aligned}$$

The last identity follows from (88) and (89), and (76) is proven. MAPLE calculates the general solution of the differential equation (76) as

$$y(x) \;=\; C_1 \cdot \frac{J_2\big(2\sqrt{b(x-1)}\big)}{x-1} + C_2 \cdot \frac{Y_2\big(2\sqrt{b(x-1)}\big)}{x-1} \tag{90}^*$$

with arbitrary constants $C_1$ and $C_2$. We now calculate the special solutions $y(x) = g_1(x)$ and $y(x) = g_2(x)$, once for $u_m = p_m$ and another time for $u_m = q_m$. In any case we obtain the two constants $C_1$ and $C_2$ from the initial conditions $g_1(0) = p_0 = 1$, $g_1'(0) = p_1 = b+2$, $g_2(0) = q_0 = 1$, and $g_2'(0) = q_1 = 2$. All results are entered into (90) and then merged to form the function $g(x) = \mathfrak{z}_0^b g_2(x) - g_1(x)$, where $\mathfrak{z}_0^b$ is expressed by (70) with $a = 0$. After significant simplifications of the resulting terms, MAPLE calculates the final result given in (77) on the right side of the formula. □

*Proof of Corollary 4.* There is nothing to prove for (84), (86), and (87), because the assertions follow directly from Theorem 5. In particular, the principal term in the asymptotic expansion of $x^{5/4}g(x)$ in (82) takes it's maximum and minimum values for $x$ satisfying

$$\frac{5\pi}{4} + 2\sqrt{b(x-1)} \;=\; \frac{\pi}{2} + k\pi \qquad (k = 1, 2, 3, \dots)\,.$$

It remains to prove (85). Let $\beta := 2\sqrt{b}$ and $z := \sqrt{1-x}$. Then, by (77),

$$E \;:=\; \int_0^{\infty} g(x)\,dx \;=\; \frac{\sqrt{b}}{I_1(\beta)} \int_0^{\infty} \frac{I_2(\beta z)}{z^2}\,dx\,. \tag{91}$$

Next, we apply formula (74) for $\mu = 1$ and so it takes the form

$$I_2(\beta z) \;=\; \frac{z}{\beta}\frac{d}{dz}\Big(\frac{I_1(\beta z)}{z}\Big)\,.$$

Moreover, we have

$$\frac{dz}{dx} \;=\; -\frac{1}{2\sqrt{1-x}} \;=\; -\frac{1}{2z}\,, \qquad \text{or} \qquad dx \;=\; -2z\,dz\,.$$

Overall, the integral in (91) can then be transformed as follows,

$$\begin{aligned} E &= \frac{\sqrt{b}}{I_1(\beta)} \int_0^{\infty} \frac{1}{\beta z}\frac{d}{dz}\Big(\frac{I_1(\beta z)}{z}\Big)\,dx \;=\; \frac{\sqrt{b}}{I_1(\beta)} \int_1^{i\infty} \frac{-2z}{\beta z}\,\frac{d}{dz}\Big(\frac{I_1(\beta z)}{z}\Big)\,dz \\ &= \frac{-1}{I_1(\beta)} \int_1^{i\infty} d\Big(\frac{I_1(\beta z)}{z}\Big) \;=\; \frac{-1}{I_1(\beta)}\Big[\frac{I_1(\beta z)}{z}\Big]_{z=1}^{z=i\infty} \\ &\overset{(72)}{=} \frac{-1}{I_1(\beta)}\Big(\lim_{t\to\infty} \frac{J_1(\beta t)}{t} - I_1(\beta)\Big) \overset{(71)}{=} \frac{-1}{I_1(\beta)} \cdot \big(-I_1(\beta t)\big) \;=\; 1\,. \end{aligned}$$

This proves the identity in (85). □

## 5. Concluding Comments

The results of this paper represent ideas born from the early correspondences of the authors. Further generalizations, to include a study of both

$$\frac{I_{a/b}(2/b)}{I_{a/b+1}(2/b)} \;=\; \big[\,\overline{a+kb}\,\big]_{k=0}^{\infty} \quad \text{and} \quad \mathfrak{z}_a^b \;=\; \sqrt{b}\frac{I_a(2\sqrt{b})}{I_{a+1}(2\sqrt{b})} = a + 1 + \frac{b|}{|a+2} + \frac{b|}{|a+3} + \cdots\,,$$

are forthcoming.

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